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A Chance Constrained Multiperiod Model for Inventory Management

by

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I am deeply appreciative to Professor A. Charnes for his energetic supervision of this paper. The novel approach to the inventory control problem developed here would not have been possible without his incredible expertise in optimization. I will always be indebted to him for the amount of time and thought he's put into supervising this paper.

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SUPERVISING PROFESSOR: A. CHARNES

We address the multi-item, multi-period inventory control problem (with stochastic customer requisitions and leadtimes for resupply) at each Air Force base level supply stores for "Budget Code 1" items subject to probabilistic constraints on a) the maximum dollar investment; b) the minimum probability of immediately meeting customer requisitions on high priority items; and c) meeting previously unfilled demands on high priority items as quickly as possible.

We develop a chance constrained program for this problem that will yield decision rules expressing the decision variables (quantities of each item

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to order at each period) that are consistent with the constraints above and which maximize the sum of the probabilities of meeting customer requisitions on all items over a specified number of periods. We then develop an equivalent deterministic program whose solution will determine the coefficients of these decision rules.

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Chapter 1

Introduction

1.1 Overview

We will address inventory control for consumable items (verses repairable items) at Air Force base-level supply stores. Our objective is to manage the inventory at the supply store so that we meet as many customer requisitions as possible for n items over N periods subject to a budget constraint. Customer requisitions and re-supply leadtimes are stochastic random variables considered stochastically independent. Further, we want to be confident of meeting customer requisitions on high priority items. We assume that requisitions we can't meet from stock on-hand can be supplied later (i.e. complete backlogging), but we want to guarantee that unfilled requisitions for high priority items will be filled as soon as possible.

In this report I'll start with a single-period model that isn't very realistic but whose structure allows solutions to be developed more easily, then move to a more complicated model which is much closer to reality. The first model will seek to minimize Expected Backorder Costs times a weighting factor set for each item.

The second model will seek to maximize the sum of the probabilities of meeting customer requisitions on n items across N periods subject to chance constraints concerning: a) dollars obligated; b) meeting requisition demands for important items; and c) meeting unfilled requisition on important items as quickly as possible.

1.2 Background on the Current Air Force System

Currently the Air Force bases use a software package called the Standard Base Supply System (SBSS) to compute the leveling parameters for each item in their consumable inventories. The SBSS sets these parameters for each item:

r_i Reorder Point–when the inventory position of item i is below r_i then an order is placed

q_i Quantity to Order–any orders placed will be for this quantity

Base-level purchases for consumable items are funded through a "stock fund." New orders can be placed to replenish the inventory only if enough funds are available in the "stock fund." When a customer "buys" an item from the base's inventory, money for the item is transferred from one of his funds (usually O&M–Operations and Maintenance) to the "stock fund."

While the formulas used to set these parameters in the SBSS are designed to reduce long-run costs for the inventory, they ignore any constraints on the solution. The parameters are computed independently for each item, based only on the items past demand pattern, cost, and estimates of ordering and holding costs. These parameters are intended to be used in a *continuous review* inventory system, i.e. the inventory level of each item i is continuously monitored and when the level drops below r_i, an order for q_i units of item i

should be placed. But, since these orders must be funded from the "stock fund," often its impossible to place this order. Consequently the parameters the SBSS computes are often unusable when daily decisions are made on what quantities of which items need to be order while staying within budget constraints.

Chapter 2

A Single Period Model

In this chapter we will consider a formulation of the Inventory Control problem for a single period of length L days. The objective will be to determine a single order-up-to parameter, y_i for each item, so that if the starting inventory is y_i for each item, then the sum over all items of Weighted Expected Backorders will be minimized for the period. These y_i must be chosen so that the value of the inventory does not exceed a budget constraint of I_{max}dollars.

2.1 Notation

L Length of Period in days

θ Random Variable for the number of customer demands for the period. [UNITS]

$f_i(\theta)$ Probability density function for the number of demands of item i for the period

c_i Unit Cost of item i [per UNIT]

I_{max} Budget Constraint in $'s for on-hand inventory

y_i VARIABLE–optimal starting on-hand inventory for item i. [UNITS]

π_i Weighting Factor for each backorder of item i [per UNIT]

$E_i(y_i)$ Expected number of backorders for item i for the period if the starting inventory is y_i [UNITS]

2.2 Assumptions

- Single Period model of length L days
- Stock is raised to y_i before beginning of period
- No ordering of stock is permitted during period
- Maximum investment in inventory is I_{max}dollars
- Demand for each item is stochastically independent of the other items' demands
- The distribution of demands for each item i can be accurately approximated by a known finite discrete distribution.

This single-period model is formulated:

$$\min \sum_{i=1}^{n} \pi_i E_i(y_i) \tag{2.1}$$

$$s.t. \quad \sum_{i=1}^{n} c_i y_i \leq V$$

$$y_i \geq 0 \; \forall i = 1, \ldots, n$$

$$y_i \text{ is an integer } \forall i = 1, \ldots, n$$

with $E_i(y_i)$ = the expected value of non-negative $(\theta - y_i)$

$$E_i(y_i) = \int_{\theta=y_i}^{\infty} (\theta - y_i) f_i(\theta) d\theta$$

2.3 A Continuous Relaxation

If we drop the constraint that the computed y_i be integers and instead treat them as continuous, our problem becomes

$$\begin{array}{lll} \min \sum_{i=1}^{n} \pi_i E_i(y_i) & & \\ s.t. \;\; \sum_{i=1}^{n} c_i y_i & \leq & V \\ \qquad\quad y_i & \geq & 0 \quad \forall i = 1, \ldots, n \end{array} \tag{2.2}$$

The expected number of backorders of item i for the period, given y_i, denoted $E_i(y_i)$ is

$$E_i(y_i) = \int_{\theta=y_i}^{\infty} (\theta - y_i) f_i(\theta) d\theta. \tag{2.3}$$

Note that $E_i$is truncated for θ values that can be met by on-hand inventory, i.e. for $\theta < y_i$.

By replacing $(\theta - y_i)$ by $\max\{0, (\theta - y_i)\}$, denoted $[\theta - y_i]^+$, we can rewrite E_i integrating over all nonnegative values of θ.

$$E_i = \int_0^{\infty} [\theta - y_i]^+ f_i(\theta) d\theta \tag{2.4}$$

Let $p_{i,\alpha}$ = Probability$\{\alpha$ units of item i are demanded$\}$. By the assumption that the demand can be approximated by a finite discrete distribution, there exists k_i for each i such that, $\sum_{\alpha=1}^{k_i} p_{i,\alpha} = 1$, then

$$E_i(y_i) = \sum_{\alpha=1}^{k_i} \left(p_{i,\alpha} * [\theta_\alpha - y_i]^+ \right)$$

So our model can be formulated

$$
\begin{aligned}
\min \sum_{i=1}^{n} & \left\{\pi_i \textstyle\sum_{\alpha=1}^{k_i} \left(p_{i,\alpha} * [\theta_\alpha - y_i]^+\right)\right\} \\
\textstyle\sum_{i=1}^{n} c_i y_i & \leq V \\
y_i & \geq 0 \quad \forall i = 1, \ldots, n
\end{aligned} \tag{2.5}
$$

With the E_i expression written in this form, the problem is still a non-linear problem (since the max function is non-linear). However, since $[\cdot]^+$ is convex, the objective function is convex.

Now, since the indices of the objective function are now constant, i.e. not dependent on the y_i, this model can be coded for optimization by any non-linear optimization package that allows non-smooth objective functions.

However, we can get better results if we use the following completely equivalent Linear Programming Model.

2.4 An Equivalent LP formulation

The only non-linear term in the objective function is $[\theta - y_i]^+$. This maximum function can rewritten

$$\max\{0, \theta - y_i\} = \frac{1}{2}\{|\theta - y_i| + (\theta - y_i)\} \tag{2.6}$$

Now, $|\theta - y_i|$ is the only nonlinear term. To replace by a linear term,

$$\text{let } \theta_\alpha - y_i - \delta^+_{i,\alpha} + \delta^-_{i,\alpha} = 0 \tag{2.7}$$

where δ's are non-negative slack variables and θ_α represents demands of α units.

Lemma 1 *$\vec{y^*}$ for the following formulation will also be optimal for (2.5).*

$$\min \sum_{i=1}^{n} \left\{ \frac{\pi_i}{2} \sum_{\alpha=1}^{k_i} p_{i,\alpha} \left[\delta^+_{i,\alpha} + \delta^-_{i,\alpha} + \theta_\alpha - y_i\right] \right\}. \tag{2.8}$$

Proof: For each i, α combination, if $p_{i,\alpha} > 0$, then at the min of this function, $\delta^+_{i,\alpha} \cdot \delta^-_{i,\alpha} = 0$. i.e. at least one of the δ's is zero. Otherwise $\exists \epsilon > 0$ such that

$$\begin{aligned} \delta^+_{i,\alpha} &= \delta^+_{i,\alpha} - \epsilon \text{ and} \\ \delta^-_{i,\alpha} &= \delta^-_{i,\alpha} - \epsilon \end{aligned}$$

satisfy condition 2.7 and produce a lower minimum value for the objective function 2.8. Since one of the δ's are zero, $|\theta_\alpha - y_i| = \delta^+_{i,\alpha} + \delta^-_{i,\alpha}$. So 2.8 is equivalent to 2.5 for i, α combinations such that $p_{i,\alpha} > 0$.

On the other hand, for i, α combinations where $p_{i,\alpha} = 0$, then

$$p_{i,\alpha} * [\delta^+_{i,\alpha} + \delta^-_{i,\alpha} + \theta - y_i] = 0 = p_{i,\alpha} * [\theta - y_i]^+.$$

and 2.8 is trivially equivalent to 2.5. $\mathcal{QED}$

So we can reformulate our initial problem as the following completely linear programming problem.

$$
\begin{array}{l}
\min \sum_{i=1}^{n} \left\{ \frac{\pi_i}{2} \sum_{\alpha=1}^{k_i} p_{i,\alpha} \left[\delta_{i,\alpha}^{+} + \delta_{i,\alpha}^{-} + \theta_\alpha - y_i \right] \right\} \\
s.t. \\
\begin{array}{rcll}
\sum_{i=1}^{n} c_i y_i & \leq & I_{max} & \\
y_i + \delta_{i,\alpha}^{+} - \delta_{i,\alpha}^{-} & = & \theta_\alpha & \text{for } \alpha = 1, \ldots, k_i \\
 & & & \text{for } i = 1, \ldots, n \\
\delta_{i,\alpha}^{+}, \delta_{i,\alpha}^{-}, y_i & \geq & 0 & \text{for } i = 1, \ldots, n
\end{array}
\end{array}
\tag{2.9}
$$

Conclusion: If we approximate our demand distribution with a finite discrete distribution, our initial non-linear formulation (2.5) can be written as the completely equivalent linear programming problem (2.9). In fact, if we compute the solution with an adjacent extreme point algorithm, then by the LIEP theorem, [1]at each iteration (basic solution), at most one of the $\delta_{i,\alpha}^{+}$,$\delta_{i,\alpha}^{-}$will be positive.

[1]Prof. Charnes' Linear Independent Extreme Points (LIEP) Theorem (1950), see [5].

Chapter 3

A Chance Constrained Multiperiod Model for Inventory Management

3.1 Problem Statement

The objective is to maximize the sum of the probabilities of filling all customer requisitions at a base supply store for n inventory items over the specified N periods through the determination for each period of (stochastic) decision rules from the class of "informationally feasible" rules linear in the requisitions. The decision variables expressed by these rules will be the quantity of each item to order for each period which will satisfy probabilistic, e.g. "chance", constraints concerning: a) dollars obligated; b) meeting requisition demands for important items; and c) unfulfilled customer requisitions for important items.

We will concentrate only on consumables items (as distinct from repairable items). The Air Force refers to the items we will concentrate on as "Budget Code 1" items of "XB3" items. These items are ordered by the base supply store and paid for from the base's "stock fund" (managed by the base).

3.2 Assumptions

- Customer requisitions (called "Sales") to be filled each period are random variables with known joint distribution functions.

- Orders in each period are placed once by the supply store.
- Time of arrival of orders ("Receipts") from time of order is stochastic, i.e. the probabilities of arrival in later periods are known. When an order for an item arrives, the quantity is exactly what was ordered. Further the stochastic arrivals of orders are assumed to be stochastically independent, both from other orders of the same item and orders of different items.
- Sales for each period that cannot be met from stock on-hand at the beginning of the period can be supplied later.
- We can identify a subset of important items for which we want to insure with high probability meeting customer requisitions.
- The cost of each item is constant throughout the N period horizon.
- Money for each order is obligated in the same period in which the order is placed. Funds for the requisitions (sales) in a period are received in the period the requisition is placed, even if the requisition is filled in a later period.

3.3 Formulation

Let

c_i [dollars per unit] Cost of each unit of item i.

I_{max} [dollars] The upper budget limit on the amount of money that can be invested in the inventory.

$S_i(t)$ [units] Customer requisitions ("Sales") of item i during period t (random variable)

$R_i(t)$ [units] Receipts of item i up through period t. These units are available to satisfy unfilled requisitions through period $t-1$ (random variable)

$Y_i(t)$ [units] Quantity of item i to order at period t (Decision Variable)

$I_i(t)$ [units] Inventory on-hand of item i at beginning of period t. This quantity is available to satisfy Sales for period t.

- for $i = 1, \ldots, n$
- for $t = 1, \ldots, N$

3.4 Formulation

We want to maximize the sum over all the N periods and over all the n items of the probabilities of satisfying requisitions ("Sales"), i.e.

$$\max \sum_{t=1}^{N} \sum_{i=1}^{n} P(I_i(t) \geq S_i(t))$$

subject to the various constraints described generally above.

These constraints contain stochastic elements both with regard to time, e.g. as in receipt of orders placed, and with regard to conformance of ordering with Air Force stockage policy.

3.5 Chance Constraints and Their Random Variables

$I_i(t)$ is a complicated function of the previous orders, receipts and sales. We suppose that individual receipts are in the same amounts as orders. However, we assume that the number of periods between placing an order and that order being available to fill customer requisitions is a discrete random variable with known probabilities. (These probabilities may be different for each item in the inventory.) If an order $Y_i(t)$ is placed its receipt cannot be used to satisfy customer requisitions (sales) until period $t+1$, even if the order is received in period t. Of course, it may not be received until period $t+1$, $t+2$, etc.

We proceed to determine the stochastic receipts of an item at a particular period in terms of the amount and period of order placement.

For $t \geq s$, define

$$p_i(t-s) = \text{Prob(Order } Y_i(s) \text{ arrives precisely in period } t).$$

It should be noted that $p_i(t-s)=0$ for $t-s \geq$ say 4.

Let $R_i(Y_i(s)\ ,t)$ be the amount received by period t of an order $Y_i(s)$ placed in period s.

Then we set

$$R_i(Y_i(s)\ ,t) \quad = \quad \begin{cases} 0, & s > t \\ Y_i(s) & \text{with probability } p_i(t-s) \text{for } s \leq t \\ & \text{where } p_i(\cdot) \text{ are known probabilities.} \end{cases}$$

Since the random variables of the arrivals of receipts (of the same or different items) are assumed stochastically independent, we can write $R_i(t)$, the cumulative receipts of item i through period t, as the sum of independent $R_i(Y_i(s)\ ,t)$ random variables,

$$R_i(t) = \sum_{s=1}^{t} R_i(Y_i(s)\ ,t)$$

In other words, since $R_i(Y_i(s)\ ,t)$ is either zero or the amount $Y_i(s)$ ordered, $R_i(t)$can only take on values that are a ım of one or more of the previous $Y_i(s)$'s.

For example,

$$R_i(1) = \begin{array}{ll} \textit{Value} & \textit{with probability} \\ \left\{ \begin{array}{l} 0, \\ Y_i(1), \end{array} \right. & \begin{array}{l} (1 - p_i(1)) \\ p_i(1) \end{array} \end{array}$$

$$R_i(2) = \begin{array}{ll} \textit{Value} & \textit{with probability} \\ \left\{ \begin{array}{l} 0, \\ Y_i(1), \\ Y_i(2), \\ Y_i(1) + Y_i(2), \end{array} \right. & \begin{array}{l} (1 - p_i(0))(1 - p_i(1)) \\ p_i(1)(1 - p_i(0)) \\ p_i(0)(1 - p_i(1)) \\ p_i(1)p_i(0) \end{array} \end{array}$$

$$R_i(3) = \begin{array}{ll} \textit{Value} & \textit{with probability} \\ \left\{ \begin{array}{l} 0 \\ Y_i(1), \\ Y_i(2), \\ Y_i(3), \\ Y_i(1) + Y_i(2), \\ Y_i(1) + Y_i(3), \\ Y_i(1) + Y_i(2), \\ Y_i(1) + Y_i(2) + Y_i(3), \end{array} \right. & \begin{array}{l} (1 - p_i(2))(1 - p_i(1))(1 - p_i(0)) \\ p_i(2)(1 - p_i(1))(1 - p_i(0)) \\ (1 - p_i(2))p_i(1)(1 - p_i(0)) \\ (1 - p_i(2))(1 - p_i(1))p_i(0) \\ p_i(2)p_i(1)(1 - p_i(0)) \\ p_i(2)(1 - p_i(1))p_i(0) \\ (1 - p_i(2))p_i(1)p_i(0) \\ p_i(2)p_i(1)p_i(0) \end{array} \end{array}$$

Let

$$z_i(t) = \prod_{j=1}^{t} (1 - p_i(j))$$

$$\tilde{p}_{i,j} = \frac{p_i(j)}{(1 - p_i(j))}$$

Then in general,

$$
R_i(t) \;=\; \begin{cases}
\textit{Value} & \textit{with probbability} \\
0 & z_i(t) \\
Y_i(1) & \tilde{p}_{i,1} z_i(t) \\
Y_i(2) & \tilde{p}_{i,2} z_i(t) \\
\vdots & \vdots \\
Y_i(t) & \tilde{p}_{i,t} z_i(t) \\
Y_i(1) + Y_i(2) & \tilde{p}_{i,1}\tilde{p}_{i,2} z_i(t) \\
Y_i(1) + Y_i(3) & \tilde{p}_{i,1}\tilde{p}_{i,3} z_i(t) \\
Y_i(2) + Y_i(3) & \tilde{p}_{i,2}\tilde{p}_{i,3} z_i(t) \\
\vdots & \vdots \\
Y_i(j) + \ldots + Y_i(k) + \ldots + Y_i(l) & \tilde{p}_{i,j} \ldots \tilde{p}_{i,k} \ldots \tilde{p}_{i,l} z_i(t) \\
\vdots & \vdots \\
Y_i(1) + Y_i(2) + \ldots + Y_i(t) & \tilde{p}_{i,1}\tilde{p}_{i,2} \ldots \tilde{p}_{i,t} z_i(t)
\end{cases}
$$

We will approximate $R_i(t)$by a random variable of Gaussian (normal) type with the same mean and variance.

Let $\overline{Y_{i,j}} = E[Y_i(j)]$ and $\overline{R_i(t)} = E[R_i(t)]$ where E denotes expectation with respect to $S_i(t)$'s, then

$$
\begin{aligned}
\overline{R_i(t)} \quad &= z_i(t)\{\overline{Y_{i,1}}\tilde{p}_{i,1} + \overline{Y_{i,2}}\tilde{p}_{i,2} + \ldots + \overline{Y_{i,t}}\tilde{p}_{i,t} \\
&+ \ldots + [\overline{Y_{i,j}} + \ldots + \overline{Y_{i,k}} + \ldots + \overline{Y_{i,l}}]\tilde{p}_{i,j}\ldots\tilde{p}_{i,k}\ldots\tilde{p}_{i,l} \\
&+ \ldots + [\overline{Y_{i,1}} + \ldots + \overline{Y_{i,t}}]\tilde{p}_{i,1}\ldots\tilde{p}_{i,t}\}
\end{aligned}
$$

and

$$
\begin{aligned}
E[R_i(t)^2] \quad &= z_i(t)\{\overline{Y_{i,1}^2}\tilde{p}_{i,1} + \overline{Y_{i,2}}\tilde{p}_{i,2} + \ldots + \overline{Y_{i,t}^2}\tilde{p}_{i,t} \\
&+ \ldots + [\overline{Y_{i,j}^2}^2 + \ldots + \overline{Y_{i,k}^2} + \ldots + \overline{Y_{i,l}^2}]\tilde{p}_{i,j}\ldots\tilde{p}_{i,k}\ldots\tilde{p}_{i,l} \\
&+ \ldots + [\overline{Y_{i,1}^2} + \ldots + \overline{Y_{i,t}^2}]\tilde{p}_{i,1}\ldots\tilde{p}_{i,t}\}
\end{aligned}
$$

From the above equations, we can calculate the variance of $R_i(t)$,

$$
\mathrm{Var}(R_i(t)) = E\left[R_i(t) - \overline{R_i(t)}\right]^2 = E[R_i(t)^2] - \overline{R_i(t)}^2
$$

The inventory available to fill sales for period t, i.e. the starting inventory for period t, denoted by $I_i(t)$ can be written

$$
I_i(t) \;= I_i(1) \;+ R_i(t-1) - \sum_{s=1}^{t-1} S_i(s)
$$

Since the quantity $I_i(1)$ is a constant and we know the distributions for $R_i(s)$ and $S_i(s)$we could obtain the exact distribution of $I_i(t)$ from the convolution of the distribution of $R_i(s)$ and $-\sum_{s=1}^{t-1} S_i(s)$. Howerver, since we make the above approximations, we need only work with the means and variances.

3.5.1 Constraints for Meeting Requisitions of Important Items

The important items that we mentioned in the assumptions section are "MIC1" items. MIC is an acronym for Mission Impact Code. If a shortage of a item has ever caused a mission to be cancelled, the item is coded as "MIC1." Other items are coded "MIC2", "MIC3", or "MIC4." We require that each period's requisitions (Sales) for MIC1 items be met with at least probability α,i.e.

$$P(I_i(t) \geq S_i(t)) \geq \alpha, \text{for } i\epsilon \text{ MIC1} \quad \text{for } t = 1, \ldots, N$$

Let $D_i(t) = \sum_{k=1}^{t} S_i(k)$, then we can rewrite the constraint for the MIC1 items,

$$P\{I_i(t) \geq S_i(t)\} \geq \alpha$$

as

$$P\{I_i(1) + R_i(t-1) - \sum_{s=1}^{t-1} S_i(s) \geq S_i(t)\} \geq \alpha$$

or

$$P\{I_i(1) + R_i(t-1) \geq D_i(t)\} \geq \alpha$$

Taking the $S_i(t)$'s to be normally distributed, employing linear stochastic decision rules and approximating the $R_i(t)$,$S_i(t)$combinations by normally distributed variates (e.g. using their means and approximating variances), we will invert these chance constraints in the process of obtaining a convex nonlinear programming problem to determine the coefficients of the optimal linear stochastic decision rules.

3.5.2 Constraints for Matching "due-ins" to "due-outs"

We call an unfilled requisition a "due-out" and each item of an outstanding order a "due-in." For MIC1 items (high priority items), we wish to have each "due-out" covered by "due-ins." So for these items, the sum of all Sales before period t should be less than or equal to the initial inventory plus all orders through the current period t, i.e.

$$\sum_{k=1}^{t-1} S_i(k) \leq I_i(1) + \sum_{k=1}^{t} Y_i(k) \quad \text{for } t = 1, \ldots, N .$$

If we let $O_i(t)$ denote the cumulative orders through period t, i.e.

$O_i(t) = \sum_{k=1}^{t} Y_i(k)$, (hence $Y_i(k) = O_i(k) - O_i(k-1)$), then we wish to have

$$D_i(t-1) \leq I_i(1) + O_i(t)$$

These probability one chance constraints requirements can be met by constraints *on the stochastic decision rules* employed for the $Y_i(k)$'s as induced by these constraints on the $O_i(t)$'s. For example, if these are to be linear,

$$O_i(t) = a_i(t)D_i(t) - b_i(t)I_i(1)$$

with constraints $a_i(t) \geq 1$ and $0 \leq b_i(t) \leq 1$ for MIC1 items will insure probability one satisfaction of these MIC1 constraints.

3.5.3 Constraints on Dollars Obligated in Each Period

Since the amount of money permitted to be invested in the inventory is limited to a budget amount, I_{max}, we must try to insure with high probability that the optimal $Y_i(t)$ values lead to investments in inventory that are less than or equal to the budget limit.

As noted earlier, dollars are obligated for each order as soon as the order is placed and funds are received for the amount of each sale (requisition) in the same period as the sale, even if we are unable to fill the sale from stock on-hand at the beginning of the period.

The money obligated for item i at period t then is

$$c_i \{I_i(1) + O_i(t) - D_i(t)\}$$

The money obligated for all items then would be

$$\sum_{i=1}^{n} c_i \{I_i(1) + O_i(t) - D_i(t)\}$$

So our dollars obligated chance constraints can be formulated as

$$P\{\sum_{i=1}^{n} [c_i(I_i(1) + O_i(t) - D_i(t))] \leq I_{max}\} \geq \phi \quad \text{for } t = 1, \ldots, N$$

where ϕ is close to one.

3.5.4 Summary of the Chance Constrained Problem

Using informationally feasible linear stochastic decision rules, assuming the $S_i(t)$are independent Gaussian random variables, and approximating the $R_i(t)$by Gaussian random variables as indicated before, we will have the following constraints and stochastic expressions to invert and/or re-express in terms of the coefficients of the stochastic decision rules. We then schematically form the convex programming problem, with linear and quadratic inequality constraints in terms of these coefficients and an objective function. Solution of this problem will determine the coefficients in the linear stochastic decision rules which will be an optimal vector of linear stochastic decision rules for this chance constrained programming problem. To summarize where we are so far in formulae,

$$\max \sum_{t=1}^{N} \sum_{i=1}^{n} P\{I_i(1) + R_i(t-1) \geq D_i(t)\}$$

$s.t.$

$$\begin{array}{rcll}
P\{I_i(1) + R_i(t-1) \geq D_i(t)\} & \geq & \alpha & \forall i \epsilon \text{ MIC1} ; \\
 & & & \text{for } t = 1, \ldots, N \\
P\{\sum_{i=1}^{n} [c_i(I_i(1) + O_i(t) - D_i(t))] \leq I_{max}\} & \geq & \phi & \text{for } t = 1, \ldots, N \\
a_i(t) D_i(t) - b_i(t) I_i(1) & = & O_i(t) & \text{for } i = 1, \ldots, n ; \\
 & & & \text{for } t = 1, \ldots, N \\
a_i(t) & \geq & 1 & \forall i \epsilon \text{ MIC1} ; \\
 & & & \text{for } t = 1, \ldots, N \\
a_i(t) & \geq & 0 & \text{for } i \not\epsilon \text{ MIC1} ; \\
 & & & \text{for } t = 1, \ldots, N \\
0 \leq b_i(t) & \leq & 1 & \forall i \epsilon \text{ MIC1} ; \\
 & & & \text{for } t = 1, \ldots, N \\
0 & \leq & b_i(t) & \text{for } i \not\epsilon \text{ MIC1} ; \\
 & & & \text{for } t = 1, \ldots, N
\end{array}$$

3.6 Inverting the Chance Constraints

Next, we invert the chance constraints to form an equivalent deterministic convex programming problem.

3.6.1 Inverting the Constraint for Meeting Important Items' Requisitions

We can rewrite:

$$P\{I_i(1) + R_i(t-1) \geq D_i(t)\} \geq \alpha$$

as

$$P\{R_i(t-1) - D_i(t) \leq I_i(1)\ \} \geq \alpha$$

From 3.5, we know the means and variances of the Gaussian random variables with which we will approximate the distributions of the $R_i(t-1)$'s. Further, since we have the mean and variance of the $S_i(t)$'s from their joint distribution (assumed known), we can easily calculate the mean, $\overline{D_i(t)}$, and variance, $\mathrm{Var}(D_i(t))$, of the $D_i(t)$'s. By the linearity of expectation, the mean of $R_i(t-1) - D_i(t)$ is then

$$\overline{R_i(t-1) - D_i(t)} = \overline{R_i(t-1)} - \overline{D_i(t)}$$

and the standard deviation, denoted $\sigma_i(t)$, is

$$\sigma_i(t) = \sqrt{\mathrm{Var}(R_i(t-1)) + \mathrm{Var}(D_i(t))}$$

Since the random variables have Gaussian or "normal" distribution, the resulting distribution is Normal with mean $\overline{R_i(t-1)} - \overline{D_i(t)}$and standard deviation $\sigma_i(t)$. To transform into a Standard Normal expression, we subtract the mean and divide through by the standard deviation in the chance constraint

$$P\{\frac{R_i(t-1) - D_i(t) - \overline{R_i(t-1)} - \overline{D_i(t)}}{\sigma_i(t)} \leq \frac{I_i(1)\ - \overline{R_i(t-1)} - \overline{D_i(t)}}{\sigma_i(t)}\} > \alpha$$

Let $\mathcal{N}(\cdot)$ denote the standard normal cumulative distribution function, then our constraint becomes

$$\mathcal{N}\left(\frac{I_i(1)\ - \overline{R_i(t-1)} - \overline{D_i(t)}}{\sigma_i(t)}\right) \geq \alpha$$

But $\mathcal{N}(\cdot)$ is a strictly one-to-one function and therefore has an inverse which we denote $\mathcal{N}^{-1}(\cdot)$. Also note that $\alpha \geq .5$ implies that $\mathcal{N}^{-1}(\alpha) \geq 0$. Applying $\mathcal{N}^{-1}(\cdot)$ to both sides, we get

$$\frac{I_i(1) - \overline{R_i(t-1)} - \overline{D_i(t)}}{\sigma_i(t)} \geq \mathcal{N}^{-1}(\alpha)$$

Since $\sigma_i(t)$must be positive, this implies

$$I_i(1) - \overline{R_i(t-1)} - \overline{D_i(t)} \geq \sigma_i(t)\mathcal{N}^{-1}(\alpha) \geq 0$$

By introducing spacer variables, $v_i(t)$, we have

$I_i(1) - \overline{R_i(t-1)} - \overline{D_i(t)} \geq v_i(t) \geq \sigma_i(t)\mathcal{N}^{-1}(\alpha) \geq 0$ and hence $I_i(1) - \overline{R_i(t-1)} - \overline{D_i(t)} - v_i(t) \geq 0$ and $v_i(t) \geq 0$ and implying

$[v_i(t)]^2 \geq [\sigma_i(t)]^2[\mathcal{N}^{-1}(\alpha)]^2$. Therefore $0 \geq [\sigma_i(t)]^2[\mathcal{N}^{-1}(\alpha)]^2 - v_i(t)^2$. (Note $[\sigma_i(t)]^2 = \mathrm{Var}(R_i(t-1) - D_i(t))$.)

Combining these results with the results for the mean and variance of $R_i(\cdot)$ from 3.5, the chance constraints for MIC1 items is inverted to the following deterministic constraints

$$\begin{aligned} v_i(t) &\geq 0 \\ I_i(1) - \overline{R_i(t-1)} - \overline{D_i(t)} - v_i(t) &\geq 0 \\ \sigma_i(t)^2[\mathcal{N}^{-1}(\alpha)]^2 - v_i(t)^2 &\leq 0 \end{aligned} \tag{3.1}$$

$\forall i \epsilon$ MIC1 ,for $t = 1, \ldots, N$.

where $\sigma_i(t)^2$ is $\mathrm{Var}(R_i(t-1) - D_i(t))$.

3.6.2 Inverting the Constraints for Dollars Obligated in Each Period

The chance constraints for dollars obligated each period,

$$P\{\sum_{i=1}^{n}[c_i(I_i(1) \ + O_i(t) - D_i(t))] \leq I_{max}\} \geq \phi \quad \text{for } t = 1, \ldots, N$$

can be rewritten as

$$P\{\sum_{i=1}^{n}[c_i(I_i(1)\ )] + \sum_{i=1}^{n}[c_i(O_i(t) - D_i(t))] \leq I_{max}\} \geq \phi$$

then

$$P\{\sum_{i=1}^{n}[c_i(O_i(t) - D_i(t))] \leq I_{max} - \sum_{i=1}^{n}[c_i(I_i(1)\ )]\} \geq \phi$$

Using our stochastic decision rule restriction $O_i(t) = a_i(t)D_i(t) - b_i(t)I_i(1)$ the constraint becomes,

$$P\{\sum_{i=1}^{n}[c_i(a_i(t)D_i(t) - b_i(t)I_i(1) \ - D_i(t))] \leq I_{max} - \sum_{i=1}^{n}[c_i(I_i(1)\ )]\} \geq \phi$$

or

$$P\{\sum_{i=1}^{n}[c_i(D_i(t)(a_i(t) - 1) - b_i(t)I_i(1)\ )] \leq I_{max} - \sum_{i=1}^{n}[c_i(I_i(1)\ )]\} \geq \phi$$

or

$$P\{\sum_{i=1}^{n}[c_i(D_i(t)(a_i(t) - 1))] \leq I_{max} + \sum_{i=1}^{n}[(b_i(t) - c_i)I_i(1)\ ]\} \geq \phi$$

By

the linearity of expectation, the mean of $\sum_{i=1}^{n}\{c_i[D_i(t)(a_i(t) - 1)]\}$, denoted $\mu(t)$, will be

$\mu(t) = \sum_{i=1}^{n}\{c_i[(ait-1]\overline{D_i(t)}\}$

where $\overline{D_i(t)}$ denotes the mean of the cumulative sales which we can obtain from the known joint distribution of sales.

Similarly by stochastic independence of $D_i(t)$ for different i,

$\mathrm{Var}(\sum_{i=1}^{n}\{c_i[D_i(t)(a_i(t)-1)]\}) = \sum_{i=1}^{n}[c_i{}^2(a_i(t)-1)^2\mathrm{Var}(D_i(t))]$

Let $\delta(t)$ represent the corresponding standard deviation. Using the Gaussian approximation to the $D_i(t)$ distribution as before so that we can reduce to standard normal expressions, we can rewrite this chance constraint as

$$P\{\frac{\sum_{i=1}^{n}[c_i(D_i(t)(a_i(t)-1))]-\mu(t)}{\delta(t)} \leq \frac{I_{max}+\sum_{i=1}^{n}[(b_i(t)-c_i)I_i(1)\ ]-\mu(t)}{\delta(t)}\} \geq \phi$$

Letting $\mathcal{N}(\cdot)$ denote the standard normal distribution, gives

$$\mathcal{N}\left(\frac{I_{max}+\sum_{i=1}^{n}[(b_i(t)-c_i)I_i(1)\ ]-\mu(t)}{\delta(t)}\right) \geq \phi$$

Using the same argument as in 3.6.1 with $\phi \geq .5$, we then invert this constraint to

$$I_{max}+\sum_{i=1}^{n}[(b_i(t)-c_i)I_i(1)\ ]-\mu(t) \geq \delta(t)\mathcal{N}^{-1}(\phi) \geq 0$$

Introducing spacer variables, $w(t)$, we get

$$I_{max}+\sum_{i=1}^{n}[(b_i(t)-c_i)I_i(1)\ ]-\mu(t) \geq w(t) \geq \delta(t)\mathcal{N}^{-1}(\phi) \geq 0$$

yielding the following equivalent deterministic constraints,

$$
\begin{array}{lcl}
w(t) & \geq & 0 \\
I_{max} + \sum_{i=1}^{n}[(b_i(t) - c_i)I_i(1)\ ] - \mu(t) - w(t) & \geq & 0 \\
\delta^2(t)[\mathcal{N}^{-1}\,(\phi)]^2 - w(t)^2 & \leq & 0
\end{array}
\tag{3.2}
$$

for $t = 1, \ldots, N$ where $\mu(t)$and $\delta^2(t)$represent the mean and variance of $\sum_{i=1}^{n}[c_i(D_i(t)(a_i(t) - 1))]$.

3.6.3 Re-expressing the Functional

The objective of the chance constrainted program, as stated before, is

$$\max \sum_{t=1}^{N} \sum_{i=1}^{n} P\{I_i(1) \ + R_i(t-1) \geq D_i(t)\}$$

We would like to maximize this sum of probabilities, but doing this would lead to a very complicated problem for which no convenient or constructive solution method is available, so instead we propose the following two surrogate expressions whose maximumizations are coherent with maximization of these probabilities.

(1) Maximize the minimum of these probabilities.

(2) Maximize the sum of the expected values of the random variables, i.e. $\max \sum_{t=1}^{N}\sum_{i=1}^{n}[I_i(1) \ + \overline{R_i(t-1)} - \overline{D_i(t)}]$ subject to conditions implying that these terms are non-negative.

Maximizing the minimum of these probabilities. If we choose to maximize the minimum of these probabilities, then in an exactly analogous manner to section 3.6.1, we rewrite

$$P\{I_i(1) \ + R_i(t-1) \geq D_i(t)\}$$

as

$$P\{\frac{R_i(t-1) - D_i(t) - \overline{R_i(t-1)} - \overline{D_i(t)}}{\sigma_i(t)} \leq \frac{I_i(1) \ - \overline{R_i(t-1)} - \overline{D_i(t)}}{\sigma_i(t)}\}$$

to get to the standard normal distribution.

Let $\gamma_i(t)$be defined such that

$$\mathcal{N}\left(\frac{I_i(1) \ -\overline{R_i(t-1)} - \overline{D_i(t)}}{\sigma_i(t)}\right) = \mathcal{N}\left(\gamma_i(t)\right)$$

so $\dfrac{I_i(1) \ -\overline{R_i(t-1)} - \overline{D_i(t)}}{\sigma_i(t)} = \gamma_i(t).$

Since $\mathcal{N}(\cdot)$ is a strictly increasing function, maximizing on the $\gamma_i(t)$will maximize on the $\mathcal{N}(\gamma_i(t))$ and hence on the probabilities of filling the requisitions when placed. Maximizing on the minimum of these probabilities can be stated as

$$\begin{aligned} &\max \gamma \\ &s.t. \\ &\qquad \gamma_i(t) \ \geq \ \gamma \quad \text{for } i = 1, \ldots, n \text{ ,for } t = 1, \ldots, N \end{aligned} \tag{3.3}$$

or

$$\begin{aligned} &\max \gamma \\ &s.t. \\ &\frac{I_i(1) \ -\overline{R_i(t-1)} - \overline{D_i(t)}}{\sigma_i(t)} \ \geq \ \gamma \quad \text{for } i = 1, \ldots, n \text{ ,} \\ &\qquad\qquad\qquad\qquad\qquad\quad \text{for } t = 1, \ldots, N \end{aligned} \tag{3.4}$$

We re-express this new constraint in terms of $a_i(\cdot), b_i(\cdot), and \gamma$ as follows.

Note $0 \leq \gamma \leq 1$ since γ represents a probability. So,

$\frac{I_i(1) - \overline{R_i(t-1)} - \overline{D_i(t)}}{\sigma_i(t)} \geq \gamma \geq 0.$

Since $\sigma_i(t) > 0$, we can rewrite this expression as

$I_i(1) - \overline{R_i(t-1)} - \overline{D_i(t)} \geq \sigma_i(t)\gamma \geq 0.$

Introducing spacer variables, $u_i(t)$, we have

$I_i(1) - \overline{R_i(t-1)} - \overline{D_i(t)} \geq u_i(t) \geq \sigma_i(t)\gamma \geq 0.$

Yielding the following deterministic constraints,

$$\begin{aligned} u_i(t) &\geq 0 \\ I_i(1) - \overline{R_i(t-1)} - \overline{D_i(t)} - u_i(t) &\geq 0 \\ \sigma_i(t)^2\gamma^2 - u_i(t)^2 &\leq 0 \end{aligned} \tag{3.5}$$

for $i \notin$ MIC1 ,for $t = 1, \ldots, N$.

where $\sigma_i(t)^2$ is $\mathrm{Var}(R_i(t-1) - D_i(t))$.

Maximizing the sum of the expected values of the original objective function. If we choose the surrogate of maximizing the sum of the expected values of the random variables, then our new objective function would be $\max \sum_{t=1}^{N} \sum_{i=1}^{n} [I_i(1) + \overline{R_i(t-1)} - \overline{D_i(t)}]$ subject to the following additional constraints to insure each these terms are non-negative $I_i(1) + \overline{R_i(t-1)} - \overline{D_i(t)} \geq 0$ for $i \notin$ MIC1

3.7 Summary of Equivalent Deterministic Convex Programming Problems

We now combine the results from section 3.6 to obtain the following convex deterministic non-linear programs. All the definitions are as before.

3.7.1 Using Surrogate Objective (1)

If we choose surrogate objective (1), to maximize the minimum of the probabilities, we obtain the following convex deterministic program.

$\max \gamma$

s.t.

$$
\begin{array}{rcll}
u_i(t) & \geq & 0 & \text{for } i \notin \text{MIC1} \\
 & & & \text{for } t = 1, \ldots, N \\
I_i(1) \; - \overline{R_i(t-1)} - \overline{D_i(t)} - u_i(t) & \geq & 0 & \text{for } i \notin \text{MIC1} \\
 & & & \text{for } t = 1, \ldots, N \\
\sigma_i(t)^2 \gamma^2 - u_i(t)^2 & \leq & 0 & \text{for } i \notin \text{MIC1} \\
 & & & \text{for } t = 1, \ldots, N \\
I_i(1) \; - \overline{R_i(t-1)} - \overline{D_i(t)} - v_i(t) & \geq & 0 & \forall i \epsilon \, \text{MIC1} \; ; \\
 & & & \text{for } t = 1, \ldots, N \\
\text{Var}(R_i(t-1) - D_i(t))[\mathcal{N}^{-1}(\alpha)]^2 - v_i(t)^2 & \leq & 0 & \forall i \epsilon \, \text{MIC1} \; ; \\
 & & & \text{for } t = 1, \ldots, N \\
v_i(t) & \geq & 0 & \forall i \epsilon \, \text{MIC1} \; ; \\
 & & & \text{for } t = 1, \ldots, N \\
I_{max} + \sum_{i=1}^{n} [(b_i(t) - c_i) I_i(1) \,] - \mu(t) & \geq & 0 & \text{for } t = 1, \ldots, N \\
\delta^2(t)[\mathcal{N}^{-1}(\phi)]^2 - w(t)^2 & \leq & 0 & \text{for } t = 1, \ldots, N \\
w(t) & \geq & 0 & \text{for } t = 1, \ldots, N \\
a_i(t) & \geq & 1 & \forall i \epsilon \, \text{MIC1} \; ; \\
 & & & \text{for } t = 1, \ldots, N \\
a_i(t) & \geq & 0 & \text{for } i \notin \text{MIC1} \; ; \\
 & & & \text{for } t = 1, \ldots, N \\
0 \leq b_i(t) & \leq & 1 & \forall i \epsilon \, \text{MIC1} \; ; \\
 & & & \text{for } t = 1, \ldots, N \\
0 & \leq & b_i(t) & \text{for } i \notin \text{MIC1} \; ; \\
 & & & \text{for } t = 1, \ldots, N
\end{array}
$$

where $\delta^2(t)$is the variance of $\sum_{i=1}^{n}[c_i(D_i(t)(a_i(t)-1))]$ and $[\sigma_i(t)]^2$ is the variance of $R_i(t-1) - D_i(t)$.

3.7.2 If we Choose the Surrogate Objective (2)

If we choose surrogate objective (2), maximizing the sum of the expected values, then we have the following convex deterministic program.

$$\max \sum_{t=1}^{N} \sum_{i=1}^{n} I_i(1) \; + \overline{R_i(t-1)} - \overline{D_i(t)}$$

s.t.

$$\begin{array}{rcll}
I_i(1) \; + \overline{R_i(t-1)} - \overline{D_i(t)} & \geq & 0 & \text{for } i \notin \text{MIC1 ;} \\
 & & & \text{for } t = 1, \ldots, N \\
I_i(1) \; - \overline{R_i(t-1)} - \overline{D_i(t)} - v_i(t) & \geq & 0 & \forall i \epsilon \text{ MIC1 ;} \\
 & & & \text{for } t = 1, \ldots, N \\
\text{Var}(R_i(t-1) - D_i(t))[\mathcal{N}^{-1}(\alpha)]^2 - v_i(t)^2 & \leq & 0 & \forall i \epsilon \text{ MIC1 ;} \\
 & & & \text{for } t = 1, \ldots, N \\
v_i(t) & \geq & 0 & \forall i \epsilon \text{ MIC1 ;} \\
 & & & \text{for } t = 1, \ldots, N \\
I_{max} + \sum_{i=1}^{n}[(b_i(t) - c_i)I_i(1) \;] - \mu(t) & \geq & 0 & \text{for } t = 1, \ldots, N \\
\delta^2(t)[\mathcal{N}^{-1}(\phi)]^2 - w(t)^2 & \leq & 0 & \text{for } t = 1, \ldots, N \\
w(t) & \geq & 0 & \text{for } t = 1, \ldots, N \\
a_i(t) & \geq & 1 & \forall i \epsilon \text{ MIC1 ;} \\
 & & & \text{for } t = 1, \ldots, N \\
a_i(t) & \geq & 0 & \text{for } i \notin \text{MIC1 ;} \\
 & & & \text{for } t = 1, \ldots, N \\
0 \leq b_i(t) & \leq & 1 & \forall i \epsilon \text{ MIC1 ;} \\
 & & & \text{for } t = 1, \ldots, N \\
0 & \leq & b_i(t) & \text{for } i \notin \text{MIC1 ;} \\
 & & & \text{for } t = 1, \ldots, N
\end{array}$$

where $\delta^2(t)$is the variance of $\sum_{i=1}^{n}[c_i(D_i(t)(a_i(t)-1))]$ and $[\sigma_i(t)]^2$ is the variance of $R_i(t-1) - D_i(t)$.

3.7.3 Conclusion

For either surrogate objective function, a convex quadratic programming problem is obtained for $a_i(\cdot), b_i(\cdot), \gamma$, and the spacer variables in these expressions. The optimal solutions for these will lead to expressions for the optimal stochastic decisions in terms of the $Y_i(t)$.

BIBLIOGRAPHY

[1] Brooke, A., Kendrick D., and Meeraus, A. (1988). *GAMS: A User's Guide.* The Scientific Press; USA.

[2] Charnes, A. and W.W. Cooper, "Chance-Constrained Programming," *Management Science*, Vol. 6, No. 1, October, 1959, pp. 73-79.

[3] Charnes, A. and W.W. Cooper, *Management Models and Industrial Applications of Linear Programming*, Vols. I and II, John Wiley & Sons, New York, 1961.

[4] Charnes, A. and W.W. Cooper, "Deterministic Equivalents for Optimizing and Satisficing Under Chance Constraints," *Operations Research*, Vol. 11, No. 1, pp. 18-39.

[5] Charnes, A., W.W. Cooper, S. Duffuaa, and M. Kress, "Complexity and Computability of Solutions to Linear Programming Systems," *International Journal of Computer and Information Sciences*, Vol. 9, No. 6, 1980, pp. 483-506.

[6] Charnes, A., W.W. Cooper, and M.J.L. Kirby, "Chance-Constrained Programming: An Extension of Statistical Method," *Optimizing Methods in Statistics*, Academic Press, Inc., New York and London, 1971, pp. 391-402.

[7] Charnes, A., and M.J.L. Kirby, "Some Special P-Models in Chance-constrained Programming," *Management Science*, Vol. 15, 1968, pp. 33-48.

[8] Hadley, G., and T.M. Whitin, *Analysis of Inventory Systems.* Prentice-Hall, Englewood Cliff, 1963.

[9] Yielding, G. *Interim Report: Demand Forecasting and the Aggregate Model* Air Force Logistics Management Center, Gunter AFB, AL (in print).